Lecture 9. ***Theorems of the second (direct) method of Lyapunov according to stability of nonlinear ACS "in big"***

Let a nonlinear ACS be described by a system of nonlinear differential equations of the following form:

$$\dot{x} = F(x_1, x_2, \ldots\ldots, x_n) \qquad (6.a)$$

$$u = f(y) \qquad (6.b)$$

$$f(0) = 0 \qquad (6.c)$$

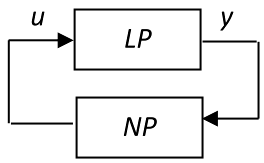

Fig.6.1. Presentation of nonlinear system

Let a linear part of the system be described by a system of differential equations in form (6.a), initial conditions *x(0) = 0* are given; the nonlinear part of the system is described by equation (6.b); equation (6.c) shows, that working area *D* contains the beginning of coordinates.

Here $x \overset{\Delta}{=} \Delta x$ is a deviation from established motion, it means that system has indignant motion from initial condition *x=0* or working mode.

Let's *V(x)* be A.M. Lyapunov's function.

THEOREM 1. If for system (6.a) there exists a fixed positive-sign function *V(x)* > 0 in area *D*, full differential of which by time $\frac{dV}{dt}$, taken according to system (6.a), is strictly negative-sign function, then indignant motion *x=0* is *asymptotically stable by Lyapunov.*

If function $V(x)$ satisfies to conditions of definition 2, i.e. function is of fixed positive-sign:

$$\begin{cases} V(x) > 0, \ if \ x \neq 0 \\ V(x) = 0, \ if \ x = 0 \end{cases},$$

it's full differential $\frac{dV}{dt}$ is of fixed negative sign, it means $\frac{dV}{dt} < 0$ if $x \neq 0,$

where $\frac{dV}{dt} \overset{\Delta}{=} (\nabla V)^T \ \frac{dx}{dt} = (\nabla V)^T \ f(x,t)\,;$

$$\nabla V \overset{\Delta}{=} \begin{bmatrix} \frac{\partial V}{\partial x_1} \\ \frac{\partial V}{\partial x_2} \\ \cdots \\ \frac{\partial V}{\partial x_n} \end{bmatrix}$$ is the gradient of scalar function *V(x)*;

f(x, t) is the right part of equation (6.a), undisturbed motion is asymptotically stable by Lyapunov, it means $\lim\limits_{t\to\infty} x(t) = 0$ (fig. 6.2).

THEOREM 2. If for system (6.a) there exists in area *D* of fixed positive-sign function *V(x)*, a full differential by time $\frac{dV}{dt}$, taken according to system (6.a), is function of constant negative-sign, then undisturbed motion *x=0* is *stable by Lyapunov.*

If $\begin{cases} V(x) > 0 \\ \frac{dV}{dt} \le 0 \end{cases}$, then undisturbed motion is *simply stable by Lyapunov* (fig. 6.2), in all of it $|x_k| \ge \alpha,\ \alpha > 0,$

where x_k is the final value, which does not reach the beginning of coordinates.

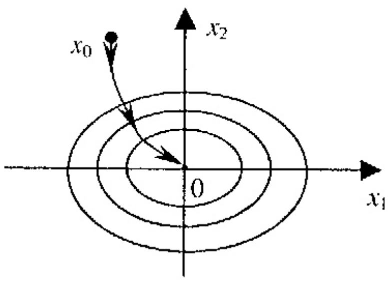

Fig. 6.2. Asymptotically stable system

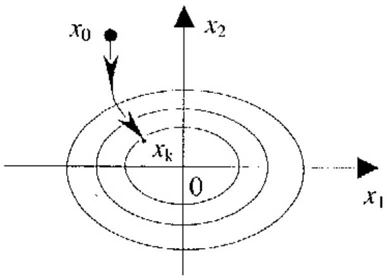

Fig. 6.3. Stable system

THEOREM 3. If for system (6.a) in area *D* there exists fixed positive-sign function *V(x)*, the full differential by time of which, taken according to the system (6.a) in some part of area *D*, which contain the beginning of coordinates, by sign coincides with the sign of function *V(x)*, then nondisturbed motion *x = 0* is *unstable by Lyapunov.*

If $\begin{cases} V(x) > 0 \\ \dfrac{dV}{dt} > 0 \end{cases}$, then by Lyapunov nondisturbed motion is *unstable* (fig. 6.4).

The task to find Lyapunov's function, which for concrete system will give necessary and sufficient condition of stability, is rather difficult.

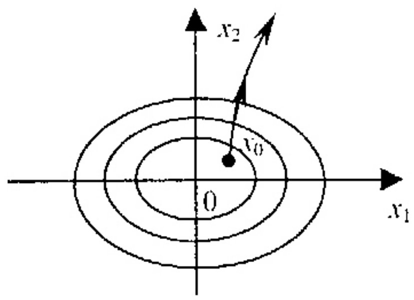

Fig. 4. An unstable system

The task about finding of function of Lyapunov which for a specific system would give *necessary and sufficient condition of stability "in big"* is very difficult. At first we will consider finding of function of Lyapunov for the linearized (linear) ACS, and then we will consider one of methods of finding of function of Lyapunov for nonlinear ACS.